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Axiomatizations of Permission Values for Games with a Hierarchical Permission Structure using Split Neutrality

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Abstract

Recently, cooperative game theory has been applied to various economic allocation problems in which players are not fully anonymous but belong to some relational structure. One of the most developed models in this respect are communications situations or (symmetric) network situations in which players can only cooperate if there are sufficient communication links in the communication network.

Another class of applications considers situations in which the players are hierarchically ordered, i.e. they are part of a structure of asymmetric relations. Examples are auctions, airport games, sequencing situations, the water distribution problem and hierarchically structured firms. This paper is about *games with permission structure* being a general game theoretic model to study situations with asymmetric relations between the players. We provide new axiomatic characterizations of the Shapley permission values and the first characterizations of the Banzhaf permission values using split properties which say something about the payoffs of players if we split certain players in two.

Keywords: Cooperative game theory, hierarchical permission structure, Shapley value, Banzhaf value, split neutrality.
JEL classification: C71

1 Introduction

A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v \colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. The collection of all characteristic functions on a particular player set N is denoted by $\mathcal{G}^N$.

In a TU-game there are no restrictions on the cooperation possibilities of the players, i.e., every coalition $E \subset N$ is feasible and can generate a payoff. Recently, TU-games are applied to economic allocation problems where there are restrictions on the possibilities of cooperation. One of the first models in this respect are the *games in coalition structure* in which the set of players is partitioned into disjoint sets which represent social groups such that for a particular player it is more easy to cooperate with players in its own group than to cooperate with players in other groups (see, e.g., Aumann and Drèze (1974), Owen (1977), Hart and Kurz (1983) and Winter (1989)).

Perhaps the most developed and applied model in which there are restrictions on the possibilities of cooperation are the *games with limited communication structure* where the edges of an undirected graph on the set of players represent binary communication links between the players such that players can cooperate only if they are connected. Following Myerson (1977), many authors studied solutions for such situations by applying well-known solutions for TU-games to the graph restricted game, i.e. TU-games in which a coalition can only earn the sum of the worths of its connected components[1]. These communication situations are widely applied in the economic networks literature that follows the seminal paper by Jackson and Wolinsky (1996). Other economic applications of TU-games with limited communication structure are sequencing games (see, e.g. Curiel (1888), Curiel, Potters, Rajendra Prasad, Tijs and Veltman (1993,1994) and Hamers (1995)) and water distribution games (see Ambec and Sprumont (2002)) which are both special cases of games which communication structure is a line-graph (see van den Brink, van der Laan and Vasil'ev (2003)). Besides viewing these problems as games with limited communication they can be seen as games in which there is some ordering among the players. In the water distribution problem, the order is given by the location of the players along the river. In the sequencing games the order is given by the positions of the players in the queue. Another type of sequencing game is discussed in Maniquet (2003) where there is no initial queue but the players are ordered according to their waiting cost.

The underlying paper considers *games with a permission structure* which are suitable to study situations where there is a hierarchical ordering of the players. These are TU-games in which the players are part of a hierarchical organization such that there are players that need permission from other players (refered to as their *predecessors*) before they are allowed to cooperate. Thus the possibilities of coalition formation are determined by the positions of the players in this so-called *permission structure*. Various assumptions can be made about how a permission structure affects the cooperation possibilities. In the *disjunctive approach*, as considered in Gilles and Owen (1994) and van den Brink (1997) it is assumed that every player needs permission from *at least one* of its predecessors before it is allowed to cooperate with other players. Alternatively, in the *conjunctive approach*, as developed in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996), it is assumed that every player needs permission

[1]See, e.g Kalai, Postlewaite and Roberts (1978) who consider the core as a solution, Owen (1986) for computing dividends in Myerson's graph restricted game. Examples of solutions that are not defined by applying solutions to Myerson's graph restricted game can be found in, e.g. Borm, Owen and Tijs (1992) and Hamiache (1999)). Greenberg and Weber (1986) consider games with line-graph communications structures in a non-tranfsferable utility setting.

from *all* its predecessors before it is allowed to cooperate with other players.

Given a game and a permission structure, in the same spirit as Myerson (1977), a modified game is defined which takes account of the limited cooperation possibilities. The conjunctive and disjunctive approach yield different modified games. A *solution* for these games is a function that assigns to every game with a permission structure a payoff distribution over the individual players. Applying solutions for TU-games (being functions that assign a payoff distribution to every TU-game) to these modified games yields solutions for games with a permission structure. Applying, for example, the *Shapley value* (Shapley (1953)) yields the *disjunctive* and *conjunctive Shapley permission values*. An alternative is to apply the *Banzhaf value* which is based on the Banzhaf index for voting games (Banzhaf (1965)) and is generalized to arbitrary games by, e.g., Owen (1975) and Dubey and Shapley (1979). This yields the *disjunctive* and *conjunctive Banzhaf permission values*.

The conjunctive and disjunctive Shapley permission values have been characterized in van den Brink and Gilles (1996) and van den Brink (1997), respectively. In van den Brink (1999) it has been shown that these two permission values only differ with respect to the *fairness* axiom that they satisfy. The disjunctive Shapley permission value satisfies *disjunctive fairness* implying that deleting the relation between a player and one of its predecessors changes the payoffs of these two players by the same amount (under the condition that the subordinate player has at least one other predecessor)[2]. Instead, the conjunctive Shapley permission value satisfies *conjunctive fairness* implying that deleting the relation between a player and one of its predecessors changes the payoffs of this player and each of its *other* predecessors by the same amount.

The underlying paper has two purposes. First we give new characterizations of the Shapley permission values using certain *split neutrality* properties. Second, we show that these new properties can be adapted to give axiomatic characterizations of the Banzhaf permission values which have not been characterized so far.

The split neutrality properties of the Shapley permission values state that, if a player in a game with permission structure 'splits' in the sense that a new player enters the game with permission structure and takes over part of the contribution or supervision of the player that is already present, then the total payoff that is distributed

[2] This property is related to fairness as introduced in Myerson (1977) for games with a limited communication structure.

among all players does not change. We distinguish between two such neutrality properties. *Vertical split neutrality* considers the situation in which the new player enters the game as a null player, and enters the permission structure as a predecessor of the player that is already present. *Horizontal split neutrality* considers the situation in which the new player enters the game as a veto player for the old player, and enters the permission structure in a similar position as the old player.

Introducing these split neutrality properties is not sufficient to characterize the permission values. Therefore, we introduce a third new property. We can look at the fairness properties discussed above as some kind of 'power split' properties. Adding a relation between two players can be seen as a split of the power over that successor. A similar requirement as with the two split neutrality properties mentioned above states that after such a power split the total sum of payoffs distributed over all players does not change. We refer to this property as *power split neutrality*.

The Banzhaf permission values do not satisfy vertical split neutrality, horizontal split neutrality and power split neutrality. However, we adapt these three properties in a way so that they are satisfied by the Banzhaf permission values. Instead of requiring that the total sum of payoffs does not change after a split, we then require that the sum of payoffs of the two players into which the former player is split does not change after the split[3]. In particular, pairwise power split neutrality implies that the payoffs of two predecessors of a player change in opposite direction if we delete its relation with one of the two predecessors. This 'opposite change' property is not satisfied by the Shapley permission values (although they satisfy it if the game is monotone).

The paper is organized as follows. In Section 2 we state some preliminaries on games with a permission structure. In Section 3 we give new axiomatic characterizations of the Shapley permission values using the split neutrality properties mentioned above. In Section 4 we give characterizations of the two Banzhaf permission values. In Section 5 we make some concluding remarks discussing some extensions, possible applications and relations with other literature. Finally, there are three appendices. Appendix A contains result on the Banzhaf value for TU-games that is used in proving some results in this paper. Appendix B contains all proofs. Appendix C shows logical independence

[3]Lehrer (1988) and Haller (1994) characterize the Banzhaf value for TU-games using *amalgamation neutrality* and *collusion neutrality*, respectively. The pairwise vertical and horizontal split neutrality axioms used here are related to a similar neutrality property which is defined in Appendix A of this paper.

of the axioms that characterize the solutions that are discussed in this paper.

2 Preliminaries: games with a permission structure

In a game with a permission structure it is assumed that players who participate in a TU-game are part of a hierarchical organization in which there are players that need permission from certain other players before they are allowed to cooperate. For a finite set of players $N \subset \mathbb{N}$ such a hierarchical organization is represented by a pair (N, S), where the mapping $S\colon N \to 2^N$ is called a *permission structure*[4] on N. The players in $S(i)$ are called the *successors* of player $i \in N$ in permission structure S. The players in $S^{-1}(i) := \{j \in N \mid i \in S(j)\}$ are called the *predecessors* of i in S. By $\widehat{S}$ we denote the *transitive closure* of the permission structure S, i.e., $j \in \widehat{S}(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in S(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = j$. The players in $\widehat{S}(i)$ are called the *subordinates* of i in S, and the players in $\widehat{S}^{-1}(i) := \{j \in N \mid i \in \widehat{S}(j)\}$ are called the *superiors* of i in S. In this paper we restrict our attention to **hierarchical** permission structures being permission structures $S\colon N \to 2^N$ that are

(i) **ayclic**, i.e., $i \notin \widehat{S}(i)$ for all $i \in N$, and

(ii) **quasi-strongly connected**, i.e., there exists an $i \in N$ such that $\widehat{S}(i) = N \setminus \{i\}$.

We denote the collection of all hierarchical permission structures on a particular player set N by $\mathcal{S}_H^N$. A triple (N, v, S) with $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$ is called a *game with a (hierarchical) permission structure*. In a hierarchical permission structure there exists a *unique* player i_0 such that $\widehat{S}(i_0) = N \setminus \{i_0\}$. Moreover, $S^{-1}(i_0) = \emptyset$ for this player. We call this player the *top-player* in the permission structure. Since we only consider hierarchical permission structures we will often refer to these simply as permission structures[5].

[4] The set $\{(i,j) \in N \times N \mid i \in N,\ j \in S(i)\}$ describes a directed graph on N.

[5] The results in this paper could be adapted for acyclic permission structures that not necessarily are quasi-strongly connected. For notational convenience we restrict attention to hierarchical permission structures.

2.1 Disjunctive and conjunctive restrictions

In the *disjunctive approach* as developed in Gilles and Owen (1994) and van den Brink (1997), it is assumed that each player needs permission from *at least one* of its predecessors before it is allowed to cooperate with other players. Consequently, a coalition is feasible if and only if every player in the coalition, except the top-player i_0, has a predecessor who also belongs to the coalition. Thus, the feasible coalitions are the ones in the set

$$\Phi^d_{N,S} := \left\{ E \subset N \,\middle|\, S^{-1}(i) \cap E \neq \emptyset \text{ or } S^{-1}(i) = \emptyset \text{ for all } i \in E \right\}.$$

Note that $E \in \Phi^d_{N,S}$ implies that $i_0 \in E$. The coalitions in $\Phi^d_{N,S}$ are called the *disjunctive autonomous* coalitions in S. The largest disjunctive autonomous subset of $E \subset N$ in $S \in \mathcal{S}^N_H$ is denoted by $\sigma^d_{N,S}(E) = \cup\{F \in \Phi^d_{N,S} \mid F \subset E\}$, and is called the *disjunctive sovereign part* of E in S. It consists of those players in E that can be reached by a directed 'permission path' starting from the top-player such that all players on this path belong to coalition E. Using this concept we can transform the characteristic function v into a modified characteristic function which takes account of the limited cooperation possibilities as determined by the permission structure as follows. Given a game with a permission structure (N, v, S), the *disjunctive restriction* of v on S is the characteristic function $r^d_{N,v,S}\colon 2^N \to \mathbb{R}$ given by $r^d_{N,v,S}(E) = v(\sigma^d_{N,S}(E))$ for all $E \subset N$.

Alternatively, in the *conjunctive approach* as developed in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996), it is assumed that each player needs permission from *all* its predecessors before it is allowed to cooperate. This implies that a coalition E is feasible if and only if for every player in the coalition it holds that all its predecessors belong to the coalition. The set of feasible coalitions in this approach thus is given by

$$\Phi^c_{N,S} := \left\{ E \subset N \,\middle|\, S^{-1}(i) \subset E \text{ for all } i \in E \right\}.$$

The coalitions in the set $\Phi^c_{N,S}$ are called the *conjunctive autonomous* coalitions in S. The largest conjunctive autonomous subset of E is denoted by $\sigma^c_{N,S}(E) = \cup\{F \in \Phi^c_{N,S} \mid F \subset E\}$, and is refered to as the *conjunctive sovereign part* of E in S. It consists of all players in E whose superiors all belong to E. Given game with permission structure (N, v, S), the *conjunctive restriction* of v on S is the characteristic function $r^c_{N,v,S}\colon 2^N \to \mathbb{R}$ given by $r^c_{N,v,S}(E) = v(\sigma^c_{N,S}(E))$ for all $E \subset N$.

Example 2.1 Consider the game with permission structure (N, v, S) with $N = \{1, 2, 3, 4\}$, $v \in \mathcal{G}^N$ given by $v(E) = 1$ if $4 \in E$, $v(E) = 0$ if $4 \notin E$, and $S \in \mathcal{S}_H^N$ given by $S(1) = \{2, 3\},\ S(2) = S(3) = \{4\},\ S(4) = \emptyset$.
The disjunctive and conjunctive restrictions of v on S are given by

$$r^d_{N,v,S}(E) = \begin{cases} 1 & \text{if } E \in \{\{1,2,4\},\ \{1,3,4\},\ \{1,2,3,4\}\} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$r^c_{N,v,S}(E) = \begin{cases} 1 & \text{if } E = \{1,2,3,4\} \\ 0 & \text{otherwise,} \end{cases}$$

respectively. □

Insert Figure 1

2.2 Permission values

A *solution* for games with a permission structure is a function f that assigns a payoff distribution $f(N, v, S) \in \mathbb{R}^N$ to every game with a permission structure (N, v, S) taking into account the limited cooperation possibilities. The *disjunctive Shapley permission value* φ^d is obtained by appyling the *Shapley value* (Shapley (1953)) to the disjunctive restricted games, while the *conjunctive Shapley permission value* φ^c is obtained by applying the Shapley value to the conjunctive restricted games, i.e.,

$$\varphi^d(N, v, S) = Sh(N, r^d_{N,v,S}) \text{ and } \varphi^c(N, v, S) = Sh(N, r^c_{N,v,S}),$$

where

$$Sh_i(N, v) = \sum_{\substack{E \subset N \\ i \in E}} \frac{(|N| - |E|)!(|E| - 1)!}{|N|!}(v(E) - v(E \setminus \{i\})) \text{ for all } i \in N.$$

Alternatively, we can apply the *Banzhaf value* to the disjunctive and conjunctive restricted games, yielding the *disjunctive Banzhaf permission value* β^d and the *conjunctive Banzhaf permission value* β^c, respectively, i.e.,

$$\beta^d(N, v, S) = B(N, r^d_{N,v,S}) \quad \text{and} \quad \beta^c(N, v, S) = B(N, r^c_{N,v,S}),$$

where

$$B_i(N,v) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ i \in E}} (v(E) - v(E \setminus \{i\})) \text{ for all } i \in N.$$

Example 2.2 The disjunctive and conjunctive Shapley permission values of the game with permission structure given in Example 2.1 are $\varphi^d(N,v,S) = \frac{1}{12}(5,1,1,5)$ and $\varphi^c(N,v,S) = \frac{1}{4}(1,1,1,1)$.
The disjunctive and conjunctive Banzhaf permission values of this game with permission structure are $\beta^d(N,v,S) = \frac{1}{8}(3,1,1,3)$ and $\beta^c(N,v,S) = \frac{1}{8}(1,1,1,1)$. □

The disjunctive Shapley permission value is axiomatized in van den Brink (1997), while axiomatizations of the conjunctive Shapley permission value can be found in van den Brink and Gilles (1996) and van den Brink (1999). The first two axioms are straightforward generalizations of efficiency and additivity of solutions for TU-games.

Axiom 2.3 (Efficiency) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, it holds that $\sum_{i \in N} f_i(N,v,S) = v(N)$.*

Axiom 2.4 (Additivity) *For every $N \subset \mathbb{N}$, $v,w \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, it holds that $f(N,v+w,S) = f(N,v,S)+f(N,w,S)$, where $(v+w) \in \mathcal{G}^N$ is defined by $(v+w)(E) = v(E)+w(E)$ for all $E \subset N$.*

Player $i \in N$ is *inessential* in game with permission structure (N,v,S) if i and all its subordinates are *null* players in (N,v), i.e., if $v(E) = v(E \setminus \{j\})$ for all $E \subset N$ and $j \in \{i\} \cup \widehat{S}(i)$. The inessential player property states that inessential players earn a zero payoff.

Axiom 2.5 (Inessential player property) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $i \in N$ is an inessential player in (N,v,S) then $f_i(N,v,S) = 0$.*

The next two axioms are stated for monotone characteristic functions. A characteristic function $v \in \mathcal{G}^N$ is *monotone* if $v(E) \leq v(F)$ for all $E \subset F \subset N$. The class of all monotone characteristic functions on N is denoted by $\mathcal{G}_M^N$. Player $i \in N$ is called *necessary* in game (N,v) if $v(E) = 0$ for all $E \subset N \setminus \{i\}$. Necessary players are 'strong' players in monotone games. The necessary player property states that necessary players in monotone games always earn the highest payoff.

Axiom 2.6 (Necessary player property) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}_M^N$ and $S \in \mathcal{S}_H^N$, if $i \in N$ is a necessary player in (N, v) then $f_i(N, v, S) \geq f_j(N, v, S)$ for all $j \in N$.*

We say that player $i \in N$ dominates player $j \in N$ 'completely' if all directed 'permission paths' from the top-player i_0 to player j contain player i. We denote the set of players that player i dominates 'completely' by $\overline{S}(i)$, i.e.,

$$\overline{S}(i) = \left\{ j \in \widehat{S}(i) \;\middle|\; \begin{array}{c} i \in \{h_1, \ldots, h_{t-1}\} \text{ for every sequence of nodes } h_1, \ldots, h_t \\ \text{such that } h_1 = i_0, h_t = j \text{ and } h_{k+1} \in S(h_k),\ k \in \{1, \ldots, t-1\} \end{array} \right\}.$$

We also define $\overline{S}^{-1}(i) = \{j \in \widehat{S}^{-1}(i) \mid i \in \overline{S}(j)\}$. Weak structural monotonicity states that a player i always earns at least as much as any of its subordinates j that it dominates 'completely' in the sense that $j \in \overline{S}(i)$.

Axiom 2.7 (Weak structural monotonicity) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}_M^N$ and $S \in \mathcal{S}_H^N$, if $i \in N$ and $j \in \overline{S}(i)$ then $f_i(N, v, S) \geq f_j(N, v, S)$.*

Weak structural monotonicity is a weaker version of *structural monotonicity* as introduced in van den Brink and Gilles (1996) which states that in a monotone game with permission structure players always earn at least as much as each of their subordinates[6].

The five axioms given above are satisfied by both the disjunctive and conjunctive Shapley permission values. These two permission values differ with respect to the fairness axiom that is used. *Disjunctive fairness* states that deleting the relation between two players h and $j \in S(h)$ (with $|S^{-1}(j)| \geq 2$) changes the payoffs of players h and j by the same amount. Moreover, also the payoffs of all players i that 'completely' dominate player h, in the sense that $i \in \overline{S}^{-1}(h)$, change by this same amount[7].

For $S \in \mathcal{S}_H^N$, $h \in N$ and $j \in S(h)$ we denote

$$S_{-(h,j)}(i) = \begin{cases} S(i) \setminus \{j\} & \text{if } i = h \\ S(i) & \text{if } i \in N \setminus \{h\}. \end{cases}$$

[6] This stronger structural monotonicity is satisfied by the conjunctive Shapley- and Banzhaf permission value but is not satisfied by the disjunctive Shapley- and Banzhaf permission value, as can be seen from Example 2.2.

[7] This property is some kind of Equal loss or gain property. Since it is related to *fairness* as introduced in Myerson (1977) for games with a limited communication structure, we refer to this property as (disjunctive) fairness. In Myerson's model fairness means that deleting a communication relation between two players in an undirected communication graph has the same effect on both their payoffs. (Note that in our fairness property we require that the successor on the relation to be deleted has at least two predecessors.)

Axiom 2.8 (Disjunctive fairness) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h \in N$ and $j \in S(h)$ with $|S^{-1}(j)| \geq 2$ then*

$$f_i(N, v, S) - f_i(N, v, S_{-(h,j)}) = f_j(N, v, S) - f_j(N, v, S_{-(h,j)}) \text{ for all } i \in \{h\} \cup \overline{S}^{-1}(h).$$

The above six axioms characterize the disjunctive Shapley permission value.

Theorem 2.9 (van den Brink (1997)) *A solution f is equal to the disjunctive Shapley permission value φ^d if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.*

The conjunctive Shapley permission value does not satisfy disjunctive fairness. However, it satisfies the alternative *conjunctive fairness* which states that deleting the relation between two players h and $j \in S(h)$ (with $|S^{-1}(j)| \geq 2$) changes the payoffs of player j and any *other* predecessor $g \in S^{-1}(j) \setminus \{h\}$ by the same amount. Moreover, also the payoffs of all players that 'completely' dominate the other predecessor g change by this same amount.

Axiom 2.10 (Conjunctive fairness) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h, j, g \in N$ are such that $h \neq g$ and $j \in S(h) \cap S(g)$ then*

$$f_i(N, v, S) - f_i(N, v, S_{-(h,j)}) = f_j(N, v, S) - f_j(N, v, S_{-(h,j)}) \text{ for all } i \in \{g\} \cup \overline{S}^{-1}(g).$$

Theorem 2.11 (van den Brink (1999)) *A solution f is equal to the conjunctive Shapley permission value φ^c if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.*

3 Axiomatizations of the Shapley permission values using split neutrality properties

In this section we provide new axiomatizations of the disjunctive and conjunctive Shapley permission values using split neutrality properties. We distinguish two neutrality properties in which a player is 'split' in two. The first one considers a situation in which a player splits in two players in a vertical line. The second considers a situation in which a player splits in two players on the same horizontal level. These splits do not 'really' affect the worths that are generated by coalitions. We now need the new player to make the old player active. Therefore we require in both cases that the total sum of payoffs that is distributed over all players does not change after a split.

Let us state these two split neutrality properties more explicitly. First, let $h \in \mathbb{N} \backslash N$ be a new player whose only task is to supervise player j. So, h will be a null player in the new game and in the permission structure he becomes the only predecessor of j and gets all previous predecessors of j as its predecessors. Then *vertical split neutrality* states that the sum of the payoffs of all players (including player h) in the new game with permission structure is equal to the sum of the payoffs of all players (excluding player h) in the original game with permission structure.

Axiom 3.1 (Vertical split neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, and $S \in \mathcal{S}_H^N$, if $h \in \mathbb{N} \setminus N$ and $j \in N$ then*

$$\sum_{i \in N \cup \{h\}} f_i(N \cup \{h\}, v^{V(h,j)}, S^{V(h,j)}) = \sum_{i \in N} f_i(N, v, S),$$

where $v^{V(h,j)} \in \mathcal{G}^{N \cup \{h\}}$ is given by $v^{V(h,j)}(E) = v(E \cap N)$ for all $E \subset N \cup \{h\}$, and $S^{V(h,j)} \in \mathcal{S}_H^{N \cup \{h\}}$ is given by

$$S^{V(h,j)}(i) = \begin{cases} \{j\} & \text{if } i = h \\ (S(i) \setminus \{j\}) \cup \{h\} & \text{if } i \in S^{-1}(j) \\ S(i) & \text{if } i \in N \setminus S^{-1}(j). \end{cases}$$

Insert Figure 2

For the second split neutrality property, suppose that player $j \in N$, with $S(j) = \emptyset$ and $S^{-1}(j) \neq \emptyset$, is split in two players in the sense that a new player h enters who gets the same predecessors as j, has no successors (similar like j), while in the new game h and j veto each other. Then *horizontal split neutrality* states that the sum of the payoffs of all players (including player h) in the new game with permission structure is equal to the sum of the payoffs of all players (excluding player h) in the original game with permission structure.

Axiom 3.2 (Horizontal split neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h \in \mathbb{N} \setminus N$ and $j \in N$ with $S(j) = \emptyset$ and $S^{-1}(j) \neq \emptyset$ then*

$$\sum_{i \in N \cup \{h\}} f_i(N \cup \{h\}, v^{H(h,j)}, S^{H(h,j)}) = \sum_{i \in N} f_i(N, v, S),$$

where $v^{H(h,j)} \in \mathcal{G}^{N \cup \{h\}}$ is given by $v^{H(h,j)}(E) = \begin{cases} v(E \setminus \{j\}) & \text{if } E \subset N \\ v(E \cap N) & \text{if } E \subset N \cup \{h\} \text{ with } h \in E, \end{cases}$

and $S^{H(h,j)} \in \mathcal{S}_H^{N \cup \{h\}}$ is given by $S^{H(h,j)}(i) = \begin{cases} \emptyset & \text{if } i = h \\ S(i) \cup \{h\} & \text{if } i \in S^{-1}(j) \\ S(i) & \text{if } i \in N \setminus S^{-1}(j). \end{cases}$

Insert Figure 3

In order to give new characterizations of the Shapley permission values we need to introduce another property. We can look at the fairness properties discussed in the previous section as some kind of power split properties. Adding the relation between h and j such that h is the predecessor and j is the successor on the relation, can be seen as a split of the power of $g \in S^{-1}(j)$ over j with player h. A similar requirement as in the previous two split neutrality properties states that after such a power split the total sum of payoffs does not change. (Of course, instead of adding a relation we can say the same if we delete a relation, as done in the fairness axioms.)

Axiom 3.3 (Power split neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h \in N$ and $j \in S(h)$ with $|S^{-1}(j)| \geq 2$ then*

$$\sum_{i \in N} f_i(N, v, S) = \sum_{i \in N} f_i(N, v, S_{-(h,j)}).$$

We can characterize the Shapley permission values using the above three new axioms. In that case we do not need efficiency. However, we still need the weaker axiom that requires efficiency only for one player games with a permission structure. Note that $S \in \mathcal{S}_H^N$ with $|N| = 1$ implies that $S(i) = \emptyset$ for $i \in N$, and thus $r^d_{N,v,S} = v$ in that case.

Axiom 3.4 (One player efficiency) *For every $N \subset \mathbb{N}$ with $|N| = 1$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, it holds that $f_i(N, v, S) = v(\{i\})$ for $i \in N$.*

Replacing efficiency in Theorem 2.9 by one player efficiency, vertical split neutrality, horizontal split neutrality and power split neutrality yields a new characterization of the disjunctive Shapley permission value.

Theorem 3.5 *A solution f is equal to the disjunctive Shapley permission value φ^d if and only if it satisfies one player efficiency, vertical split neutrality, horizontal split neutrality, power split neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.*

The proof of this theorem (as well as the proofs of all further results) can be found in Appendix B of this paper. Logical independence of the axioms is shown in Appendix C.
Replacing disjunctive fairness by conjunctive fairness in Theorem 3.5 characterizes the conjunctive Shapley permission value. (Since the proof of this theorem goes along similar lines as the proof of Theorem 3.5 it is omitted.)

Theorem 3.6 *A solution f is equal to the conjunctive Shapley permission value φ^c if and only if it satisfies one player efficiency, vertical split neutrality, horizontal split neutrality, power split neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.*

4 Axiomatizations of the Banzhaf permission values

Since the Banzhaf value for TU-games is not efficient it is not surprising that the (disjunctive and conjunctive) Banzhaf permission values also do not satisfy efficiency. However, the characterizations of the Shapley permission values given in the previous section only use one-player efficiency. This weaker version of efficiency is satisfied by the Banzhaf permission values. In this section we show how the neutrality axioms can be adapted to give a characterization of the Banzhaf permission values.

Vertical split neutrality stated that the total sum of payoffs that is distributed among the players does not change if we let a player $h \in \mathbb{N} \setminus N$ who is not yet in the game enter the game as a null player and enter the permission structure as the unique predecessor of a player $j \in N$ who is already in the game. So, player h does not add anything to the generation of value and he only supervises player j. Usually, supervisors are useful only if they supervise more than one successor[8]. Therefore it seems reasonable that such a split does not generate more payoffs for these two players. In other words, the payoff for player h should go at the cost of player j, implying that the sum of the payoffs of players h and j in the new game with permission structure is the same as the payoff of player j in the original game with permission structure.

Axiom 4.1 (Pairwise vertical split neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h \in \mathbb{N} \setminus N$ and $j \in N$ then*

$$f_j(N \cup \{h\}, v^{V(h,j)}, S^{V(h,j)}) + f_h(N \cup \{h\}, v^{V(h,j)}, S^{V(h,j)}) = f_j(N, v, S),$$

where $v^{V(h,j)} \in \mathcal{G}^{N \cup \{h\}}$ and $S^{V(h,j)} \in \mathcal{S}_H^{N \cup \{h\}}$ are as given in Axiom 3.1.

Similarly, it seems reasonable that the sum of the payoffs of players h and j in the new game with permission structure is equal to the payoff of player j in the original game with permission structure after we let player h take over part of the role of player j on a horizontal level.

Axiom 4.2 (Pairwise horizontal split neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, and $S \in \mathcal{S}_H^N$, if $h \in \mathbb{N} \setminus N$ and $j \in N$ is such that $S(j) = \emptyset$ and $S^{-1}(j) \neq \emptyset$ then*

$$f_j(N \cup \{h\}, v^{H(h,j)}, S^{H(h,j)}) + f_h(N \cup \{h\}, v^{H(h,j)}, S^{H(h,j)}) = f_j(N, v, S),$$

where $v^{H(h,j)} \in \mathcal{G}^{N \cup \{h\}}$ and $S^{H(h,j)} \in \mathcal{S}_H^{N \cup \{h\}}$ are as given in Axiom 3.2.

Neutrality axioms for TU-games are introduced in Lehrer (1988) and Haller (1994) who characterize the Banzhaf value for TU-games using what they call *amalgamation neutrality* and *collusion neutrality*, respectively. The pairwise vertical and horizontal split neutrality axioms used here are related to a similar neutrality property which is defined in Appendix A of this paper.

Disjunctive and conjunctive fairness compare the effects of deleting the relation between players h and $j \in S(h)$ on the payoffs of players h and j, and on the payoffs of players $g \in S^{-1}(j) \setminus \{h\}$ and j, respectively. These properties do not compare the change in payoffs of the two predecessors h and $g \in S^{-1}(j) \setminus \{h\}$ after deleting the

[8] In the economic literature managers in firms usually have tasks such as, for example, coordinating, monitoring, or information processing. All these roles are useful only if a manager has at least two successors.

relation between h and j. It seems reasonable that these changes are opposite in sign. The disjunctive and conjunctive Shapley permission values do not satisfy this property (see Example 4.6).

It turns out that the Banzhaf permission values satisfy the 'opposite change' property for every game with a hierarchical permission structure, i.e., the disjunctive and conjunctive Banzhaf permission values of players h and g, $h \neq g$, always change in opposite direction after deleting the relation between players h and $j \in S(h) \cap S(g)$. Moreover, the absolute values of the changes in the Banzhaf permission values of these two players are the same. In other words, the sum of the payoffs of players h and g does not change. Therefore, this can be seen as a pairwise version of power split neutrality.

Axiom 4.3 (Pairwise power split neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h, g, j \in N$ are such that $h \neq g$ and $j \in S(h) \cap S(g)$ then*

$$f_h(N, v, S) + f_g(N, v, S) = f_h(N, v, S_{-(h,j)}) + f_g(N, v, S_{-(h,j)}).$$

Replacing in Theorems 3.5 and 3.6 vertical split neutrality, horizontal split neutrality and power split neutrality by the above three pairwise axioms yields characterizations of the disjunctive and conjunctive Banzhaf permission values, respectively.

Theorem 4.4 *A solution f is equal to the disjunctive Banzhaf permission value β^d if and only if it satisfies one player efficiency, pairwise vertical split neutrality, pairwise horizontal split neutrality, pairwise power split neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.*

Again, the proof can be found in Appendix B. A similar axiomatization of the conjunctive Banzhaf permission value can be given by replacing disjunctive fairness by conjunctive fairness.

Theorem 4.5 *A solution f is equal to the conjunctive Banzhaf permission value β^c if and only if it satisfies one player efficiency, pairwise vertical split neutrality, pairwise horizontal split neutrality, pairwise power split neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.*

We already mentioned before that the Shapley permission values do not satisfy the property that deleting the relation between players h and $j \in S(h) \cap S(g)$, $h \neq g$, changes the payoffs of h and g in opposite direction. We illustrate this with the following example.

Example 4.6 Consider the game with permission structure (N, v, S) given by $N = \{1,2,3,4,5\}$, $v = u_{\{4,5\}} - \frac{7}{10}u_{\{4\}}$, where u_T denotes the unanimity game of $T \subset N$ (see equation (2) in Appenndix B with $c_T = 1$), and $S(1) = \{2,3,5\}$, $S(2) = S(3) = \{4\}$ and $S(4) = S(5) = \emptyset$. In this example, $\varphi^d_2(N, v, S) - \varphi^d_2(N, v, S_{-(2,4)}) = -\frac{1}{120} - 0 < 0$ and $\varphi^d_3(N, v, S) - \varphi^d_3(N, v, S_{-(2,4)}) = -\frac{1}{120} - \frac{1}{60} = -\frac{1}{40} < 0$. Thus, deleting the relation between players 2 and 4 changes the disjunctive Shapley permission values of player 4's predecessors 2 and 3 in the same direction. (By disjunctive fairness the change in the disjunctive Shapley permission value of player 4 is the same as for player 2: $\varphi^d_4(N, v, S) - \varphi^d_4(N, v, S_{-(2,4)}) = \frac{1}{120} - \frac{1}{60} = -\frac{1}{120}$.)

For the conjunctive Shapley permission values $\varphi^c_2(N, v, S) - \varphi^c_2(N, v, S_{-(2,4)}) = \frac{1}{40} - 0 > 0$ and $\varphi^c_3(N, v, S) - \varphi^c_3(N, v, S_{-(2,4)}) = \frac{1}{40} - \frac{1}{60} = \frac{1}{120} > 0$. Thus, deleting the relation between players 2 and 4 also changes the conjunctive Shapley permission values of player 4's predecessors 2 and 3 in the same direction. (By conjunctive fairness the change in the conjunctive Shapley permission value of player 4 is the same as for player 3: $\varphi^c_4(N, v, S) - \varphi^c_4(N, v, S_{-(2,4)}) = \frac{1}{40} - \frac{1}{60} = \frac{1}{120}$.) $\square$

However, for monotone games with a hierarchical permission structure the disjunctive and conjunctive Shapley permission values satisfy the 'opposite change' property. This follows from the following proposition (which straightforward proof is omitted).

Proposition 4.7 *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}^N_H$ and $h, g, j \in N$ with $h \neq g$ and $j \in S(h) \cap S(g)$:*

1. $\varphi^d_j(N, v, S) \geq \varphi^d_j(N, v, S_{-(h,j)})$, $\varphi^d_h(N, v, S) \geq \varphi^d_h(N, v, S_{-(h,j)})$ *and* $\varphi^d_g(N, v, S) \leq \varphi^d_g(N, v, S_{-(h,j)})$

2. $\varphi^c_j(N, v, S) \leq \varphi^c_j(N, v, S_{-(h,j)})$, $\varphi^c_h(N, v, S) \geq \varphi^c_h(N, v, S_{-(h,j)})$ *and* $\varphi^c_g(N, v, S) \leq \varphi^c_g(N, v, S_{-(h,j)})$.

5 Concluding remarks

In this paper we gave new axiomatic characterizations of the disjunctive- and conjunctive Shapley permission values using split neutrality properties. Also for the first time characterizations of the disjunctive- and conjunctive Banzhaf permission values are given. All new axiomatic characterizations use nine logically independent axioms[9]. According to these characterizations, the difference between the disjunctive and conjunctive permission values is with respect to the fairness axiom that is used. According to the disjunctive permission values the payoff of a successor and one of its predecessos changes by the same amount if we delete the arc between these two players, while according to the conjunctive permission values the payoff of the successor and each other predecessor changes by the same amount. The main difference between the Shapley permission values and Banzhaf permission values is with respect to how the payoffs react to particular changes in the game and permission structure that reflect a split of contributions in the game and/or authority in the permission structure. The sum of the Shapley permission values over all players do not change, while the sum of the Banzhaf permission values of the two players that represent the original player is equal to the payoff of the original player in the original game with permission structure.

With the axioms for solutions for games with a permission structure that are formulated so far we can, by looking at different combinations of axioms, find other solutions or obtain impossibility results. For example, there is no solution that satisfies efficiency, the necessary player property, weak structural monotonicity and pairwise vertical split neutrality. A next step then could be to weaken these axioms to find new solutions.

Also, future research can be directed to generalizations and special cases. Games with a permission structure have been generalized in, e.g Algaba, Bilbao, van den Brink and Jiménez-Losada (2000, 2003a,b) to games with limited cooperation in the sense that the set of feasible coalitions forms an antimatroid. A more specific approach is to concentrate on hierarchical permission structures that have a tree structure. These structures have important applications in economic theory, in particular hierarchically

[9]Without logical independence the fewer axioms the better. However, since we also show logical independence of the axioms the more the better. Sometimes it is wrongly claimed that less (stronger) axioms is better than few (weaker) axioms. We could easily obtain less axioms by taking axioms together. For example, vertical and horizontal split neutrality can easily be formulated into one stronger axiom.

structured firms (see van den Brink (1996)) which are games with a permission structure such that the permission structure is a tree and the game is defined on the 'lowest level' in the tree, i.e. the set of players that have no successors. Other special cases are *peer group games* studied in Brânzei, Fragnelli and Tijs (2002) which essentially are games with a permission structure in which the permission structure is a tree and only one-player coalitions have a non-zero dividend in the game. So, the difference between hierarchically structured firms and peer group games is that in hierarchically structured firms the game is defined on the lowest level of the hierarchy but any game on that set of players is allowed, while in peer group games only one-player coalitions have a non-zero dividend but any player can have a non-zero dividend. Special cases include airport games and auction games (see Brânzei, Fragnelli and Tijs (2002)).

Further future research can be directed to economic applications of cooperative game theory that can be found in recent literature as already mentioned in the introduction. In the water distribution problem of Ambec and Sprumont (2002) agents are located along a river from upstream to downstream, and water flows into the river between each pair of agents. Each agent can consume its own water inflow, but can also decide to let water stream through to downstream agents. Depending on the utilities of the different agents for water, efficiency gains can be realized if an agent does not consume all its water. Main question then is how agents that let water flow downstream should be compensated for not consuming their water. In the sequencing games of Curiel (1985), Curiel et all. (1993, 1994) and Hamers (1985), a set of jobs is in a queue to be processed on one machine. Depending on the wating costs and processing times of each job, cost avings can be realized by jobs switching positions. Making switches so that we obtain an efficient queue (i.e. a queue that minimizes total costs), the main question then is how jobs that switch to positions later in the queue have to be compensated. Van den Brink, van der Laan and Vasil'ev (2003) study both examples by considering them as (undirected) line-graph games. Although the related sequencing game of Maniquet (2003) is not a line-graph game, also there the jobs are hierarchically ordered. He does not assume an initial order of the players, but looks for a fair allocation of the waiting cost (that can be compensated by monetary transfers) depending only on the waiting cost of the players. The order is given by the waiting cost of the jobs.

We remark that in order to analyze the above two mentioned applications of the water distribution problem and sequencing situations we need to generalize the games with permission structure as discussed in this paper to allow for 'non-transitivities'. In games with a permission structure, feasible coalitions containing player i always contain the top player and intermediate players between player i and the top. To analyze the water distribution and sequencing situations we first need to relax this assumption to avoid higher ranked players to earn also payoffs from cooperation of coalitions that only contain players that are lower in the hierarchy. The games as discussed in this paper then can be seen as a special case of transitive permission structures.

Last but not least we mention the network literature. We already mentioned in the introduction that perhaps the most developed field of games with limited cooperation in economics are the games with limited communication which is widely applied in the economic literature on network formation that follows the seminal paper by Jackson and Wolinsky (1996). Building on Myerson (1977)'s model with given communication structure they formulate a strategic model in which the players themselves decide what communication links to build. They extensively study the efficiency and stability of the resulting networks. In a similar way, future research can be directed to the formation of directed (or hierarchical) networks by extending games with a permission structure in a similar way as done for communication networks by Jackson and Wolinsky. In this respect the fairness and split neutrality axioms are of particular interest since they say something about the effects of adding or deleting arcs from the network.

Appendix A

In this appendix we introduce a similar neutrality property for TU-games as amalgamation and collusion neutrality as defined in Lehrer (1988) and Haller (1994), respectively. Suppose that player $h \in \mathbb{N} \setminus N$ is entering the game $v \in \mathcal{G}^N$ as a veto player for player $j \in N$. Then the sum of the Banzhaf values of players h and j in the new game is equal to the Banzhaf value of player j in the original game. To be more precize, for game (N, v) and players $j \in N$ and $h \in \mathbb{N} \setminus N$, let the characteristic function $v_{hj} \in \mathcal{G}^{N \cup \{h\}}$ be given by

$$v_{hj}(E) = \begin{cases} v(E \setminus \{j\}) & \text{if } E \subset N \\ v(E \cap N) & \text{if } E \subset N \cup \{h\},\ h \in E. \end{cases} \tag{1}$$

Proposition A.1 If $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $j \in N$ and $h \in \mathbb{N} \setminus N$ then

$B_h(N \cup \{h\}, v_{hj}) + B_j(N \cup \{h\}, v_{hj}) = B_j(N, v)$, where $v_{hj} \in \mathcal{G}^{N \cup \{h\}}$ is given by (1).

Proof. If $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $j \in N$ and $h \in \mathbb{N} \setminus N$, then $v_{hj}(E) - v_{hj}(E \setminus \{h\}) = 0$ if $j \notin E$, and $v_{hj}(E) - v_{hj}(E \setminus \{j\}) = 0$ if $h \notin E$. Thus

$$B_h(N \cup \{h\}, v_{hj}) + B_j(N \cup \{h\}, v_{hj}) =$$

$$= \frac{1}{2^{|N|}} \left(\sum_{\substack{E \subset N \cup \{h\} \\ h \in E}} (v_{hj}(E) - v_{hj}(E \setminus \{h\})) + \sum_{\substack{E \subset N \cup \{h\} \\ j \in E}} (v_{hj}(E) - v_{hj}(E \setminus \{j\})) \right)$$

$$= \frac{1}{2^{|N|}} \left(\sum_{\substack{E \subset N \cup \{h\} \\ \{h,j\} \subset E}} (v_{hj}(E) - v_{hj}(E \setminus \{h\})) + \sum_{\substack{E \subset N \cup \{h\} \\ \{h,j\} \subset E}} (v_{hj}(E) - v_{hj}(E \setminus \{j\})) \right)$$

$$= \frac{1}{2^{|N|}} \sum_{\substack{E \subset N \cup \{h\} \\ \{h,j\} \subset E}} (v(E \cap N) - v((E \cap N) \setminus \{j\}) + v(E \cap N) - v((E \cap N) \setminus \{j\}))$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ j \in E}} (v(E) - v(E \setminus \{j\})) = B_j(N, v).$$

□

Refering to the property described above as pairwise split neutrality for TU-games, to verify that the Banzhaf value for TU-games is characterized by one-player efficiency, the null player property, symmetry, additivity and pairwise split neutrality goes along the same lines as a similar result is shown in Haller (1994). The Shapley value for TU-games does not satisfy pairwise split neutrality but it satisfies the analogous property which states that the sum of the payoffs of all players in the game v_{hj} (including player h) is equal to the sum of payoffs of all players in the original game v (excluding player h). Refering to this property as split neutrality it is easy to verify that the Shapley value for TU-games is characterized by one-player efficiency, the null player property, symmetry, additivity and split neutrality.

Appendix B: Proofs

This appendix contains the proofs of the results presented in this paper. We first prove necessity of the axioms in Theorem 3.5.

Lemma B.1 The disjunctive Shapley permission value φ^d satisfies one player efficiency, vertical split neutrality, horizontal split neutrality, power split neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.

PROOF.
Since $v^{V(h,j)}(N) = v^{H(h,j)}(N) = v(N)$ for every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$, $j \in N$ and $h \in \mathbb{N} \setminus N$, φ^d satisfying efficiency implies that it satisfies one player efficiency, vertical split neutrality, horizontal split neutrality and power split neutrality. φ^d satisfying the other axioms follows directly from Theorem 2.9. $\square$

We prove the uniqueness part of Theorem 3.5 in three steps. First, we prove a lemma for positively scaled unanimity games with a permission tree that has no inessential players. We denote by $\mathcal{S}_{tree}^N = \{S \in \mathcal{S}_H^N \mid \text{ for all } i \in N \text{ it holds that } |S^{-1}(i)| \leq 1\}$ the class of all permission *trees* on N. For $T \subset N$, $T \neq \emptyset$, and $c_T > 0$, the *positively scaled unanimity game* $w_T = c_T u_T$ is given by

$$w_T(E) = \begin{cases} c_T & \text{if } E \supset T \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

For $S \in \mathcal{S}_H^N$ and $h \in N$ the permission structure $S_{-h} \in S^{N \setminus \{h\}}$ is given by

$$S_{-h}(i) = \begin{cases} (S(i) \setminus \{h\}) \cup S(h) & \text{if } i \in S^{-1}(h) \\ S(i) & \text{if } i \in N \setminus (\{h\} \cup S^{-1}(h)). \end{cases} \tag{3}$$

Insert Figure 4

Lemma B.2 If solution f satisfies one player efficiency, vertical split neutrality, horizontal split neutrality, power split neutrality, the necessary player property and weak structural monotonicity, then $f(N, w_T, S)$ is uniquely determined for all $S \in \mathcal{S}^N_{tree}$ and $w_T = c_T u_T$ with $c_T > 0$ and $T \subset N$ satisfying $T \cup \widehat{S}^{-1}(T) = N$.

PROOF.
Suppose that solution f satisfies the six axioms. Consider a hierarchical permission (tree) structure $S \in \mathcal{S}^N_{tree}$ and monotone characteristic function $w_T = c_T u_T$, $T \subset N$, $c_T > 0$ with $T \cup \widehat{S}^{-1}(T) = N$. (So, there are no inessential players.) Clearly, f satisfying the necessary player property implies that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(N, w_T, S) = c^*$ for all $i \in T$, and $f_i(N, w_T, S) \leq c^*$ for all $i \in N \setminus T$. But then weak structural monotonicity and the fact that $\widehat{S}(i) = \overline{S}(i)$ for all $i \in N$ and $S \in \mathcal{S}^N_{tree}$ imply that

$$f_i(N, w_T, S) = c^* \text{ for all } i \in T \cup \overline{S}^{-1}(T) = T \cup \widehat{S}^{-1}(T) = N. \tag{4}$$

So, we have uniquely determined $f(N, w_T, S)$ if we determine c^*. We do this by induction on $|N|$.

If $|N| = 1$ then one player efficiency implies that $f_i(N, w_T, S) = c_T$ for $i \in N$.

Proceeding by induction assume that $f_i(N', w_{T'}, S') = \frac{c_{T'}}{|N'|}$ for all $(N', w_{T'}, S')$ with $S' \in \mathcal{S}^{N'}_{tree}$, $c_{T'} > 0$, $T' \cup (\widehat{S'})^{-1}(T') = N'$ and $|N'| < |N|$.

We distinguish the following three cases (of which at least one must occur).

(i) Suppose there exist $h, g, j \in N$, $h \neq g$, such that $S(h) = S(j) = \emptyset$ and $\{h, j\} \subset S(g)$. (Note that this case can only occur if $|N| \geq 3$.) By the assumption that $T \cup \widehat{S}^{-1}(T) = N$, we have $\{h, j\} \subset T$. Since $c_T u_T = (c_T u_{T \setminus \{h\}})^{H(h,j)}$ and $S = (S_{-h})^{H(h,j)}$ with $v^{H(h,j)}$ and $S^{H(h,j)}$ as given in Axiom 3.2, horizontal split neutrality implies that

$$\sum_{i \in N} f_i(N, w_T, S) = \sum_{i \in N \setminus \{h\}} f_i(N \setminus \{h\}, c_T u_{T \setminus \{h\}}, S_{-h}). \tag{5}$$

With the induction hypothesis it follows that $\sum_{i \in N \setminus \{h\}} f_i(N \setminus \{h\}, c_T u_{T \setminus \{h\}}, S_{-h}) = (|N| - 1)\frac{c_T}{|N|-1} = c_T$. With (5) and (4) this yields that $|N| c^* = c_T$, and thus $f_i(N, w_T, S) = c^* = \frac{c_T}{|N|}$ for all $i \in N$.

(ii) Suppose that there exist $j \in T$ and $h \in N \setminus T$ with $S(j) = \emptyset$ and $S(h) = \{j\}$. Since $w_T = (w_T)^{V(h,j)}$ and $S = (S_{-h})^{V(h,j)}$ with $v^{V(h,j)}$ and $S^{V(h,j)}$ as given in Axiom 3.1, vertical split neutrality implies that

$$\sum_{i \in N} f_i(N, w_T, S) = \sum_{i \in N \setminus \{h\}} f_i(N \setminus \{h\}, w_T, S_{-h}). \tag{6}$$

Similarly as in case (i), with the induction hypothesis and (4) this yields that $|N|c^* = c_T$, and thus $f_i(N, w_T, S) = c^* = \frac{c_T}{|N|}$ for all $i \in N$.

(iii) Suppose that there exist $h, j \in T$ with $S(j) = \emptyset$ and $S(h) = \{j\}$. Then take a $g \in \mathbb{N} \setminus N$, and define $S', S'' \in \mathcal{S}^{N\cup\{g\}}$ by $S' = S^{V(j,h)}$ and

$$S''(i) = \begin{cases} \{h, j\} & \text{if } i = g \\ (S(i) \setminus \{h\}) \cup \{g\} & \text{if } i \in S^{-1}(h) \\ S(i) & \text{if } i \in N \setminus S^{-1}(h) \end{cases}$$

Insert Figure 5

(Note that $S'' \notin \mathcal{S}^{N\cup\{g\}}_{tree}$ since h and g are both predecessor of player j in S''.) Since $T \cup \overline{S''}^{-1}(T) = N \cup \{g\}$, the necessary player property and weak structural monotonicity imply that there is a $c^{**} \in \mathbb{R}$ such that

$$f_i(N \cup \{g\}, w_T, S'') = c^{**} \text{ for all } i \in N \cup \{g\}. \tag{7}$$

Since $((N \cup \{g\}) \setminus \{h\}, c_T u_{T\setminus\{h\}}, S''_{-h}) \in \mathcal{S}^{(N\cup\{g\})\setminus\{h\}}_{tree}$ (with $|(N \cup \{g\} \setminus \{h\})| = |N|$) is as considered in case (ii) it follows from that case that

$$f_i((N \cup \{g\}) \setminus \{h\}, c_T u_{T\setminus\{h\}}, S''_{-h}) = \frac{c_T}{|N|} \text{ for all } i \in (N \cup \{g\}) \setminus \{h\}. \tag{8}$$

But then, by $S''_{-(h,j)} = (S''_{-h})^{H(h,j)}$, horizontal split neutrality implies that $\sum_{i\in N\cup\{g\}} f_i(N\cup\{g\}, w_T, S''_{-(h,j)}) = \sum_{i\in(N\cup\{g\})\setminus\{h\}} f_i((N\cup\{g\})\setminus\{h\}, c_T u_{T\setminus\{h\}}, S''_{-h}) = c_T$. With the necessary player property, weak structural monotonicity and the fact that $T \cup \overline{S''_{-(h,j)}}^{-1}(T) = N \cup \{g\}$ this yields

$$f_i(N \cup \{g\}, w_T, S''_{-(h,j)}) = \frac{c_T}{|N| + 1} \text{ for all } i \in N \cup \{g\}. \tag{9}$$

Further, power split neutrality implies that

$$\sum_{i\in N\cup\{g\}} f_i(N \cup \{g\}, w_T, S'') = \sum_{i\in N\cup\{g\}} f_i(N \cup \{g\}, w_T, S''_{-(h,j)}), \tag{10}$$

which with (7) and (9) gives

$$f_i(N \cup \{g\}, w_T, S'') = \frac{c_T}{|N|+1} \text{ for all } i \in N \cup \{g\}. \tag{11}$$

Since $S' = S''_{-(g,j)}$, again applying power split neutrality yields

$$\sum_{i \in N \cup \{g\}} f_i(N \cup \{g\}, w_T, S') = \sum_{i \in N \cup \{g\}} f_i(N \cup \{g\}, w_T, S''), \tag{12}$$

which with the necessary player property, weak structural monotonicity, $T \cup \overline{S}^{-1}(T) = N \cup \{g\}$ and (11) yields

$$f_i(N \cup \{g\}, w_T, S') = \frac{c_T}{|N|+1} \text{ for all } i \in N \cup \{g\}. \tag{13}$$

Finally, vertical split neutrality yields

$$\sum_{i \in N} f_i(N, w_T, S) = \sum_{i \in N \cup \{g\}} f_i(N \cup \{g\}, w_T, S') = c_T, \tag{14}$$

and thus with (4) we have $f_i(N, w_T, S) = c^* = \frac{c_T}{|N|}$ for all $i \in N$.

So, for $S \in \mathcal{S}^N_{tree}$ and $w_T = c_T u_T$ with $c_T > 0$ and $T \cup \widehat{S}^{-1}(T) = N$, we conclude that $f_i(N, w_T, S) = c^* = \frac{c_T}{|N|}$ for all $i \in T \cup \widehat{S}^{-1}(T) = N$. □

The next step is to show that adding the inessential player property and disjunctive fairness to the axioms implies that f is uniquely determined for all positively scaled unanimity games with a permission tree.

Lemma B.3 If solution f satisfies one player efficiency, vertical split neutrality, horizontal split neutrality, power split neutrality, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness, then $f(N, w_T, S)$ is uniquely determined whenever $S \in \mathcal{S}^N_{tree}$ and $w_T = c_T u_T$ for some $T \subset N$, $T \neq \emptyset$, and $c_T > 0$.

Proof.
Suppose that solution f satisfies the eight axioms. Consider a permission (tree) structure $S \in \mathcal{S}^N_{tree}$ and monotone characteristic function $w_T = c_T u_T$ as given in (2) for some $c_T > 0$. Denote $\alpha_S(T) = T \cup \widehat{S}^{-1}(T)$.

Since all players in $N \setminus \alpha_S(T)$ are inessential players in (N, w_T, S), the inessential player property implies that $f_i(N, w_T, S) = 0$ for all $i \in N \setminus \alpha_S(T)$. Further, f satisfying the necessary player property and weak structural monotonicity and the fact that $\widehat{S}(i) = \overline{S}(i)$ for all $i \in N$ and $S \in \mathcal{S}^N_{tree}$, imply that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(N, w_T, S) = c^*$ for all $i \in \alpha_S(T)$. So,

$$f_i(N, w_T, S) = \begin{cases} c^* & \text{if } i \in \alpha_S(T) = T \cup \widehat{S}^{-1}(T) \\ 0 & \text{if } i \in N \setminus \alpha_S(T). \end{cases} \tag{15}$$

In particular, $f_{i_0}(N, w_T, S) = c^*$. So, we have determined $f(N, w_T, S)$ if we determine c^*. We do this by induction on $|N \setminus \alpha_S(T)|$.

If $|N \setminus \alpha_S(T)| = 0$ then $c^* = \frac{c_T}{|N|} = \frac{c_T}{|\alpha_S(T)|}$ is uniquely determined by Lemma B.2.

Proceeding by induction assume that $c^* = f_{i_0}(N', w_{T'}, S')$ is uniquely determined for all $(N', w_{T'}, S')$ with $S' \in \mathcal{S}^{N'}_{tree}$, $c_{T'} > 0$ and $|N' \setminus \alpha_{S'}(T')| < |N \setminus \alpha_S(T)|$.
Since $N \setminus \alpha_S(T) \neq \emptyset$ there exists a $j \in N \setminus \alpha_S(T)$ with $S(j) = \emptyset$. We distinguish the following three cases (of which at least one must occur).

(i) Suppose that $\alpha_S(T) \neq \{i_0\}$ and $j \notin \widehat{S}(h)$ for all $h \in \alpha_S(T) \setminus \{i_0\}$. Take $h \in \alpha_S(T) \setminus \{i_0\}$. Define $S' \in \mathcal{S}^N_H$ by

$$S'(i) = \begin{cases} \{h\} & \text{if } i = j \\ S(i) & \text{otherwise.} \end{cases}$$

(Note that $S' \notin \mathcal{S}^N_{tree}$ since player h has two predecessors in S'.) Also define $S'' \in \mathcal{S}^N_H$ by

$$S''(i) = \begin{cases} S'(i) \setminus \{h\} & \text{if } h \in S(i) \\ S'(i) & \text{otherwise.} \end{cases}$$

Insert Figure 6

(Note that $S'' \in \mathcal{S}^N_{tree}$.) Since $|N \setminus \alpha_{S''}(T)| < |N \setminus \alpha_S(T)|$, the induction hypothesis implies that all $f_i(N, w_T, S'')$, $i \in N$, are known. Since $S = S'_{-(j,h)}$

and $S'' = S'_{-(g,h)}$ for $g \in S^{-1}(h)$, power split neutrality and the induction hypothesis imply that

$$\sum_{i \in N} f_i(N, w_T, S) = \sum_{i \in N} f_i(N, w_T, S') = \sum_{i \in N} f_i(N, w_T, S'') \tag{16}$$

is known. With (15) the $|\alpha_S(T)|$ unkown payoffs $f_i(N, w_T, S) = c^*$, $i \in \alpha_S(T)$, are then uniquely determined.

(ii) Suppose that $\alpha_S(T) \neq \{i_0\}$ and there is an $h \in \alpha_S(T) \setminus \{i_0\}$ with $j \in \widehat{S}(h)$. Define $S' \in \mathcal{S}_H^N$ by

$$S'(i) = \begin{cases} S(i) \cup \{j\} & \text{if } i = i_0 \\ S(i) & \text{otherwise.} \end{cases}$$

Insert Figure 7

(Note that $S' \notin \mathcal{S}_{tree}^N$ since player j has two predecessors in S'.) Also define $S'' \in \mathcal{S}_H^N$ by

$$S''(i) = \begin{cases} S'(i) \setminus \{j\} & \text{if } j \in S(i) \\ S'(i) & \text{otherwise.} \end{cases}$$

(Note that $S'' \in \mathcal{S}_{tree}^N$.) Since $S = S'_{-(i_0,j)}$ and j is an inessential player in S and S', disjunctive fairness and the inessential player property imply that $f_{i_0}(N, w_T, S') - f_{i_0}(N, w_T, S) = f_j(N, w_T, S') - f_j(N, w_T, S) = 0$. Thus $f_{i_0}(N, w_T, S') = f_{i_0}(N, w_T, S)$.

Since $S'' = S'_{-(g,j)}$ for $g \in S^{-1}(j)$, disjunctive fairness, the inessential player property and $g \in \overline{S'}(i_0) \cap \overline{S''}(i_0)$ also imply that $f_{i_0}(N, w_T, S') - f_{i_0}(N, w_T, S'') = f_j(N, w_T, S') - f_j(N, w_T, S'') = 0$. Thus $f_{i_0}(N, w_T, S') = f_{i_0}(N, w_T, S'')$.

So, $f_{i_0}(N, w_T, S) = f_{i_0}(N, w_T, S') = f_{i_0}(N, w_T, S'')$. Since S'' is as in case (i) we have uniquely determined $c^* = f_{i_0}(N, w_T, S)$.

(iii) Finally, suppose that $\alpha_S(T) = \{i_0\}$. Consider the game with permission structure $(N \cup \{g\}, w_T, S^{V(g,i_0)})$. Vertical split neutrality and the inessential player property imply that $f_{i_0}(N \cup \{g\}, w_T, S^{V(g,i_0)}) + f_g(N \cup \{g\}, w_T, S^{V(g,i_0)})) = f_{i_0}(N, w_T, S)$. The necessary player property and weak structural monotonicity imply that $f_{i_0}(N \cup \{g\}, w_T, S^{V(g,i_0)})) = f_g(N \cup \{g\}, w_T, S^{V(g,i_0)}))$. Thus $f_{i_0}(N, w_T, S) = 2f_{i_0}(N \cup \{g\}, w_T, S^{V(g,i_0)}))$. Since $(N \cup \{g\}, w_T, S^{V(g,i_0)}))$ is as considered in case (ii), we determined $c^* = f_{i_0}(N, w_T, S) = \frac{1}{2} f_{i_0}(N \cup \{g\}, w_T, S^{V(g,i_0)}))$.

So, in all three cases we uniquely determined c^* and thus, with (15) we uniquely determined $f(N, w_T, S)$. $\square$

Finally, by adding additivity to the axioms of Lemma B.3, we prove the main result of Section 3.

Proof of Theorem 3.5.
With Lemma B.1 we only have to show that there can be at most one solution that satisfies the nine axioms stated in the theorem. Therefore, suppose that solution f satisfies the nine axioms.

Consider the hierarchical permission structure $S \in \mathcal{S}_H^N$ and the monotone characteristic function $w_T = c_T u_T$, $c_T \geq 0$, as given in (2). If $c_T = 0$ then the inessential player property implies that $f_i(N, w_T, S) = 0$ for all $i \in N$.

Now suppose that $c_T > 0$. Again, we denote by $\alpha_S(T) = T \cup \widehat{S}^{-1}(T)$ the set consisting of all players in T and all their superiors. For $S \in \mathcal{S}_H^N \setminus \mathcal{S}_{tree}^N$ there is at least one $i \in N$ with $\widehat{S}_i^{-1}(T) \neq \overline{S}_i^{-1}(T)$. Therefore, by $\gamma_S(T) := \{i \in \alpha_S(T) \mid T \cap (\{i\} \cup \overline{S}(i)) \neq \emptyset\}$ we denote the set of those players in $\alpha_S(T)$ who belong to T or have subordinates in T that they dominate 'completely'.

Again, the inessential player property implies that $f_i(N, w_T, S) = 0$ for all $i \in N \setminus \alpha_S(T)$. Further, f satisfying the necessary player property and weak structural monotonicity implies that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(N, w_T, S) = c^*$ for all $i \in \gamma_S(T)$. We prove that c^* and all $f_i(N, w_T, S)$, $i \in \alpha_S(T) \setminus \gamma_S(T)$, are uniquely determined by induction on the number $\sum_{i \in N} |S(i)|$.

If $\sum_{i \in N} |S(i)| = |N| - 1$ then $S \in \mathcal{S}_{tree}^N$ and $f(N, w_T, S)$ is uniquely determined by Lemma B.3. (Note that $\sum_{i \in N} |S(i)| \geq |N| - 1$ for all $S \in \mathcal{S}_H^N$.)

Proceeding by induction assume that $f(N, w_T, S')$ is uniquely determined for all $S' \in \mathcal{S}_H^N$ with $\sum_{i\in N} |S'(i)| < \sum_{i\in N} |S(i)|$.

Next we recursively define the sets L_k, $k \in \{0\} \cup \mathbb{N}$, by

$$L_0 := \emptyset, \text{ and}$$

$$L_k := \left\{ i \in N \setminus \bigcup_{t=1}^{k-1} L_t \;\middle|\; S(i) \subset \bigcup_{t=1}^{k-1} L_t \right\}, \text{ for all } k \in \mathbb{N}.$$

In van den Brink and Gilles (1994) it is shown that for hierarchical permission structures there exists an $M < \infty$ such that the sets $L_1, \ldots, L_M$ form a partition of N consisting of non-empty sets only.

Next we describe a procedure which determines the values $f_i(N, w_T, S)$ as linear functions of the constant c^*, i.e. we determine the values c_i, $i \in N$ such that

$$f_i(N, w_T, S) = c^* + c_i, \text{ for all } i \in N \tag{17}$$

Step 1 For every $i \in L_1$ one of the following two conditions is satisfied:

(i) If $i \in N \setminus \alpha_S(T)$ then $f_i(N, w_T, S) = 0$ as mentioned before. Thus $c_i = -c^*$.

(ii) If $i \in \alpha_S(T)$ then $i \in T$ since $S(i) = \emptyset$. Thus $f_i(N, w_T, S) = c^*$, i.e. $c_i = 0$.

Let $k = 2$. Goto Step 2.

Step 2 If $L_k = \emptyset$ then Stop.

Else, for every $i \in L_k$ one of the following three conditions is satisfied:

(i) If $i \in N \setminus \alpha_S(T)$ then $f_i(N, w_T, S) = 0$, and thus $c_i = -c^*$.

(ii) If $i \in \gamma_S(T)$ then $f_i(N, w_T, S) = c^*$, and thus $c_i = 0$.

(iii) If $i \in \alpha_S(T) \setminus \gamma_S(T)$ then by definition of $\alpha_S(T)$ and $\gamma_S(T)$ there exists an $h \in \{i\} \cup \overline{S}(i)$ and a $j \in S(h)$ such that $|S^{-1}(j)| \geq 2$. Disjunctive fairness implies that

$$f_i(N, w_T, S) - f_i(N, w_T, S_{-(h,j)}) = f_j(N, w_T, S) - f_j(N, w_T, S_{-(h,j)}).$$

Using the induction hypothesis and the fact that $j \in \widehat{S}(i)$ implies that we already determined c_j for which $f_j(N, w_T, S) = c^* + c_j$ (since $j \in L_l$ with

$l < k$), it follows that we have determined $c_i = c_j + f_i(N, w_T, S_{-(h,j)}) - f_j(N, w_T, S_{-(h,j)})$ such that

$$f_i(N, w_T, S) = f_j(N, w_T, S) - f_j(N, w_T, S_{-(h,j)}) + f_i(N, w_T, S_{-(h,j)}) = c^* + c_i.$$

Step 3 Let $k = k + 1$. Goto Step 2.

Since there exists an $M < \infty$ such that the sets $L_1, \ldots, L_M$ form a partition of N consisting of non-empty sets only, the procedure described above determines the values $c_i,\ i \in N$, and thus with (17) we determined all $f_i(N, w_T, S) = c^* + c_i$ as linear functions of c^* with c_i known for all $i \in N$. To determine c^*, we distinguish the following two cases.

(i) Suppose there exists $j \in \alpha_S(T)$ with $|S^{-1}(j)| \geq 2$. Take $h \in S^{-1}(j)$. Power split neutrality implies that

$$\sum_{i \in N} f_i(N, w_T, S) = \sum_{i \in N} f_i(N, w_T, S_{-(h,j)}).$$

The induction hypothesis implies that $\sum_{i \in N} f_i(N, w_T, S_{-(h,j)})$ is known. So, $\sum_{i \in N} f_i(N, w_T, S)$ is known. With (17) c^* is uniquely determined.

(ii) Suppose that $|S^{-1}(j)| = 1$ for all $j \in \alpha_S(T) \setminus \{i_0\}$. Then, by assumption, there exists a $j \in N \setminus \alpha_S(T)$ with $|S^{-1}(j)| \geq 2$. Take $h \in S^{-1}(j)$. Disjunctive fairness and the inessential player property imply that

$$f_{i_0}(N, w_T, S) - f_{i_0}(N, w_T, S_{-(h,j)}) = f_j(N, w_T, S) - f_j(N, w_T, S_{-(h,j)}) = 0.$$

So, $c^* = f_{i_0}(N, w_T, S) = f_{i_0}(N, w_T, S_{-(h,j)})$ is uniquely determined by the induction hypothesis.

In both cases we determined c^*. Since we already determined all $c_i,\ i \in N$, with (17) we then uniquely determined all values $f_i(N, w_T, S),\ i \in N$.

Now, let $S \in \mathcal{S}_H^N$ and consider the characteristic function $w_T = c_T u_T$ with $c_T < 0$. (Note that we cannot apply the necessary player property and weak structural monotonicity to this game with permission structure, because $c_T u_T$ is not monotone if $c_T < 0$.) Let $v_0 \in \mathcal{G}^N$ denote the *null game*, i.e., $v_0(E) = 0$ for all $E \subset N$. From the inessential player property it follows that $f_i(N, v_0, S) = 0$ for all $i \in N$. Since $r^d_{N,w_T,S} + r^d_{N,-w_T,S} = r^d_{N,v_0,S}$ for all $T \subset N$, additivity of f implies that $f(N, w_T, S) = f(N, v_0, S) - f(N, -w_T, S) = -f(N, -w_T, S)$. Since $-w_T = -c_T u_T$ with $-c_T > 0$ is monotone, $f(N, -w_T, S)$ is uniquely determined. Thus also $f(N, w_T, S) = -f(N, -w_T, S)$ is uniquely determined if $c_T < 0$.

Finally, since every characteristic function $v \in \mathcal{G}^N$ can be expressed as a linear combination of unanimity games $v = \sum_{T \subset N} \Delta_{(N,v)}(T) u_T$ with $\Delta_{(N,v)}(T) = \sum_{F \subset T} (-1)^{|T|-|F|} v(F)$ being the Harsanyi dividends (see Harsanyi (1959)), it follows with additivity of f that $f(N, v, S)$ is uniquely determined for every $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N_H$. □

Proof of Theorem 4.4.

Proving that β^d satisfies additivity, the inessential player property, the necessary player property and weak structural monotonicity is along the same lines as this is shown for φ^d in van den Brink (1997). One player efficiency is evident.

Pairwise vertical split neutrality of β^d follows from Proposition A.1 (see Appendix A) and the fact that $r^d_{N,v^{V(h,j)},S^{V(h,j)}} = (r^d_{N,v,S})_{hj}$ where $v^{V(h,j)}$ and $S^{V(h,j)}$ are as given in Axiom 3.1, and v_{hj} is as given in equation (1) of the appendix. Similarly, pairwise horizontal split neutrality of β^d follows from Proposition A.1 and the fact that $r^d_{N,v^{H(h,j)},S^{H(h,j)}} = (r^d_{N,v,S})_{hj}$ where $v^{H(h,j)}$ and $S^{H(h,j)}$ are as given in Axiom 3.2.

To prove that β^d satisfies pairwise power split neutrality, let $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}^N_H$ and $h, g, j \in N$ be such that $j \in S(h) \cap S(g)$, $h \neq g$. Since $\sigma^d_{N,S}(E) = \sigma^d_{N,S_{-(h,j)}}(E)$ if $h \notin E$ or $g \in E$ (where $\sigma^d_{N,S}(E)$ denotes the sovereign part of coalition E in S, see page 5), it follows that

$$\beta^d_h(N, v, S) - \beta^d_h(N, v, S_{-(h,j)}) = B_h(N, r^d_{N,v,S}) - B_h(N, r^d_{N,v,S_{-(h,j)}})$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ h \in E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S}(E \setminus \{h\})) - v(\sigma^d_{N,S_{-(h,j)}}(E)) + v(\sigma^d_{N,S_{-(h,j)}}(E \setminus \{h\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ h \in E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S_{-(h,j)}}(E)) \Big) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ h \in E \\ g \notin E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S_{-(h,j)}}(E)) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ g \notin E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S_{-(h,j)}}(E)) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ g \in E}} \Big(v(\sigma^d_{N,S}(E \setminus \{g\})) - v(\sigma^d_{N,S_{-(h,j)}}(E \setminus \{g\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ g \in E}} \Big(v(\sigma^d_{N,S_{-(h,j)}}(E)) - v(\sigma^d_{N,S_{-(h,j)}}(E \setminus \{g\})) - v(\sigma^d_{N,S}(E)) + v(\sigma^d_{N,S}(E \setminus \{g\})) \Big)$$

$$= B_g(N, r^d_{N,v,S_{-(h,j)}}) - B_g(N, r^d_{N,v,S}) = \beta^d_g(N, v, S_{-(h,j)}) - \beta^d_g(N, v, S),$$

showing that β^d satisfies pairwise power split neutrality.

To show that β^d satisfies disjunctive fairness, let $S \in \mathcal{S}^N_H$ and $h, j \in N$ be such that $j \in S(h)$ and $|S^{-1}(j)| \geq 2$. Further, let $i \in \{h\} \cup \overline{S}^{-1}(h)$.
Since $\{i, j\} \not\subset E$ implies that $\sigma^d_{N,S}(E) = \sigma^d_{N,S_{-(h,j)}}(E)$, it follows that

$$\beta^d_i(N, v, S) - \beta^d_i(N, v, S_{-(h,j)}) = B_i(N, r^d_{N,v,S}) - B_i(N, r^d_{N,v,S_{-(h,j)}})$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ i \in E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S}(E \setminus \{i\})) - v(\sigma^d_{N,S_{-(h,j)}}(E)) + v(\sigma^d_{N,S_{-(h,j)}}(E \setminus \{i\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ i \in E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S_{-(h,j)}}(E)) \Big) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ \{i,j\} \subset E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S_{-(h,j)}}(E)) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ j \in E}} \Big(v(\sigma^d_{N,S}(E)) - v(\sigma^d_{N,S}(E \setminus \{j\})) - v(\sigma^d_{N,S_{-(h,j)}}(E)) + v(\sigma^d_{N,S_{-(h,j)}}(E \setminus \{j\})) \Big)$$

$$== B_j(N, r^d_{N,v,S}) - B_j(N, r^d_{N,v,S_{-(h,j)}}) = \beta^d_j(N, v, S) - \beta^d_j(N, v, S_{-(h,j)}),$$

showing that β^d satisfies disjunctive fairness.

Uniqueness of β^d follows along the same lines as the proof of the uniqueness of φ^d. We describe the differences compared to the proofs of Lemma's B.2 and B.3 and Theorem 3.5 in characterizing β^d.
Lemma B.2 now states that: If the solution f satisfies one player efficiency, pairwise vertical split neutrality, pairwise horizontal split neutrality, pairwise power split neutrality, the necessary player property and weak structural monotonicity, then $f(N, w_T, S)$ is uniquely determined for all $S \in \mathcal{S}^N_{tree}$ and $w_T = c_T u_T$ for some $c_T > 0$ and $T \subset N$ satisfying $T \cup \widehat{S}^{-1}(T) = N$.

In the proof of Lemma B.2 the induction hypothesis now will be that $f_i(N', w_{T'}, S') = \frac{c_{T'}}{2^{|N'|-1}}$ for all $(N', w_{T'}, S')$ with $S' \in \mathcal{S}^{N'}_{tree}$, $c_{T'} > 0$, $T' \cup (\widehat{S'})^{-1}(T') = N'$ and $|N'| < |N|$. Using the pairwise versions of vertical split neutrality, horizontal split neutrality and power split neutrality we adapt the three cases that are distinguished on pages 13-14 as follows.

(i) In case (i) by using pairwise horizontal split neutrality we replace equation (5) by

$$f_j(N, w_T, S) + f_h(N, w_T, S) = f_j(N \setminus \{h\}, c_T u_{T\setminus\{h\}}, S_{-h}). \tag{18}$$

With the induction hypothesis it then follows that $f_j(N \setminus \{h\}, c_T u_{T\setminus\{h\}}, S_{-h}) = \frac{c_T}{2^{|N|-2}}$. With (4) and (18) it then also follows that $2c^* = \frac{c_T}{2^{|N|-2}}$, and thus $f_i(N, w_T, S) = c^* = \frac{c_T}{2^{|N|-1}}$ for all $i \in N$.

(ii) We arrive at a similar conclusion in case (ii) by using pairwise vertical split neutrality and replacing equation (6) by

$$f_j(N, w_T, S) + f_h(N, w_T, S) = f_j(N \setminus \{h\}, w_T, S_{-h}).$$

(iii) In case (iii), we consider the same S' and S'', and also arrive at equation (7). Then we replace equation (8) by

$$f_i((N \cup \{g\}) \setminus \{h\}, c_T u_{T\setminus\{h\}}, S''_{-h}) = \frac{c_T}{2^{|N|-1}} \text{ for all } i \in (N \cup \{g\}) \setminus \{h\}.$$

Now pairwise horizontal split neutrality implies that $f_j(N \cup \{g\}, w_T, S''_{-(h,j)}) + f_h(N \cup \{g\}, w_T, S''_{-(h,j)}) = f_j((N \cup \{g\}) \setminus \{h\}, c_T u_{T\setminus\{h\}}, S''_{-h}) = \frac{c_T}{2^{|N|-1}}$, and thus with the necessary player property and weak structural monotonicity we can replace equation (9) by

$$f_i(N \cup \{g\}, w_T, S''_{-(h,j)}) = \frac{c_T}{2^{|N|}} \text{ for all } i \in N \cup \{g\}.$$

Further, using pairwise power split neutrality instead of power split neutrality we replace equations (10), (11), (12) and (13), respectively, by

$$f_h(N \cup \{g\}, w_T, S'') + f_g(N \cup \{g\}, w_T, S'') = f_h(N \cup \{g\}, w_T, S''_{-(h,j)}) + f_g(N \cup \{g\}, w_T, S''_{-(h,j)})$$

$$f_i(N \cup \{g\}, w_T, S'') = \frac{c_T}{2^{|N|}} \text{ for all } i \in N \cup \{g\}$$

$$f_h(N \cup \{g\}, w_T, S') + f_g(N \cup \{g\}, w_T, S') = f_h(N \cup \{g\}, w_T, S'') + f_g(N \cup \{g\}, w_T, S'')$$

$$f_i(N \cup \{g\}, w_T, S') = \frac{c_T}{2^{|N|}} \text{ for all } i \in N \cup \{g\},$$

Finally, pairwise vertical split neutrality yields that (14) is replaced by

$$f_h(N, w_T, S) = f_h(N \cup \{g\}, w_T, S') + f_g(N \cup \{g\}, w_T, S') = \frac{c_T}{2^{|N|-1}}.$$

Thus with (4) we have $f_i(N, w_T, S) = c^* = \frac{c_T}{2^{|N|-1}}$ for all $i \in N$.

Lemma B.3 is now adapted to the lemma which states that: If solution f satisfies one player efficiency, pairwise vertical split neutrality, pairwise horizontal split neutrality, pairwise power split neutrality, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness, then $f(N, w_T, S)$ is uniquely determined whenever $S \in \mathcal{S}^N_{tree}$ and $w_T = c_T u_T$ for some $T \subset N$, $T \neq \emptyset$, and $c_T > 0$.
The proof is identical except for part (i) in the proof by induction to determine c^* on page 16. Now pairwise power split neutrality implies that

$$f_j(N, w_T, S) + f_{i_0}(N, w_T, S) = f_j(N, w_T, S') + f_{i_0}(N, w_T, S')$$

and

$$f_j(N, w_T, S') + f_{i_0}(N, w_T, S') = f_j(N, w_T, S'') = f_{i_0}(N, w_T, S'')$$

which with (15) and the induction hypothesis yields that $c^* = f_i(N, w_T, S),\ i \in \alpha_S(T)$, is known.
(Part (ii) is identical. In part (iii) pairwise vertical split neutrality and the inessential player property imply the same as vertical split neutrality and the inessential player property.)

In the proof of uniqueness for any game with a hierarchical permission structure, in a similar way as in the proof of Theorem 3.5 we arrive at (17) and uniquely determine all $c_i,\ i \in N$. The determination of c^* in case (i) on pages 19-20 should be replaced by the following.

(i) Suppose there exists $j \in \alpha_S(T)$ with $|S^{-1}(j)| \geq 2$. Then take $h, g \in N$, $h \neq g$, such that $j \in S(h) \cap S(g)$. Pairwise power split neutrality implies that

$$f_h(N, w_T, S) + f_g(N, w_T, S) = f_h(N, w_T, S_{-(g,j)}) + f_g(N, w_T, S_{-(g,j)}),$$

which with the induction hypothesis yields that $f_h(N, w_T, S) + f_g(N, w_T, S)$ is known. Then with (17) c^* is uniquely determined.

The remainder of the proof is the same as the proof of Theorem 3.5. □

Proof of Theorem 4.5.

Proving uniqueness and proving that β^c satisfies additivity, pairwise vertical split neutrality, pairwise horizontal split neutrality, the inessential player property, the necessary player property and weak structural monotonicity is along the same lines as this is shown for β^d. So, we only have to show that β^c satisfies pairwise power split neutrality and conjunctive fairness.

To show that β^c satisfies pairwise power split neutrality, let $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h, g, j \in N$ be such that $j \in S(h) \cap S(g)$, $h \neq g$. Since $\sigma^c_{N,S}(E) = \sigma^c_{N,S_{-(h,j)}}(E)$ if $h \in E$ or $g \notin E$, it follows in a similar way as done for β^d in the proof of Theorem 4.4 that

$$\beta^c_h(N, v, S) - \beta^c_h(N, v, S_{-(h,j)}) = B_h(N, r^c_{N,v,S}) - B_h(N, r^c_{N,v,S_{-(h,j)}})$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ h \in E}} \Big(v(\sigma^c_{N,S}(E)) - v(\sigma^c_{N,S}(E \setminus \{h\})) - v(\sigma^c_{N,S_{-(h,j)}}(E)) + v(\sigma^c_{N,S_{-(h,j)}}(E \setminus \{h\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ h \in E}} \Big(v(\sigma^c_{N,S_{-(h,j)}}(E \setminus \{h\})) - v(\sigma^c_{N,S}(E \setminus \{h\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ \{h,g\} \subset E}} \Big(v(\sigma^c_{N,S_{-(h,j)}}(E \setminus \{h\})) - v(\sigma^c_{N,S}(E \setminus \{h\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ g \in E, h \notin E}} \Big(v(\sigma^c_{N,S_{-(h,j)}}(E)) - v(\sigma^c_{N,S}(E)) \Big) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ g \in E}} \Big(v(\sigma^c_{N,S_{-(h,j)}}(E)) - v(\sigma^c_{N,S}(E)) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ g \in E}} \Big(v(\sigma^c_{N,S_{-(h,j)}}(E)) - v(\sigma^c_{N,S_{-(h,j)}}(E \setminus \{g\})) - v(\sigma^c_{N,S}(E)) + v(\sigma^c_{N,S}(E \setminus \{g\})) \Big)$$

$$= B_g(N, r^c_{N,v,S_{-(h,j)}}) - B_g(N, r^c_{N,v,S}) = \beta^c_g(N,v,S_{-(h,j)}) - \beta^c_g(N,v,S),$$

showing that β^c satisfies pairwise power split neutrality.

To show that β^c satisfies conjunctive fairness, let $S \in \mathcal{S}^N_H$ and $h,g,j \in N$ be such that $j \in S(h) \cap S(g)$, $h \neq g$. Further, let $i \in \{g\} \cup \overline{S}^{-1}(g)$. The proof is similar to the corresponding proof for disjunctive fairness of the disjunctive Banzhaf permission value, but with the roles of h and g exchanged. Since $\{i,j\} \not\subset E$ implies that $\sigma^c_{N,S}(E) = \sigma^c_{N,S_{-(h,j)}}(E)$, it follows that

$$\beta^c_i(N,v,S) - \beta^c_i(N,v,S_{-(h,j)}) = B_i(N, r^c_{N,v,S}) - B_i(N, r^c_{N,v,S_{-(h,j)}})$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ i \in E}} \Big(v(\sigma^c_{N,S}(E)) - v(\sigma^c_{N,S}(E \setminus \{i\})) - v(\sigma^c_{N,S_{-(h,j)}}(E)) + v(\sigma^c_{N,S_{-(h,j)}}(E \setminus \{i\})) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ i \in E}} \Big(v(\sigma^c_{N,S}(E)) - v(\sigma^c_{N,S_{-(h,j)}}(E)) \Big) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ \{i,j\} \subset E}} \Big(v(\sigma^c_{N,S}(E)) - v(\sigma^c_{N,S_{-(h,j)}}(E)) \Big)$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ j \in E}} \Big(v(\sigma^c_{N,S}(E)) - v(\sigma^c_{N,S}(E \setminus \{j\})) - v(\sigma^c_{N,S_{-(h,j)}}(E)) + v(\sigma^c_{N,S_{-(h,j)}}(E \setminus \{j\})) \Big)$$

$$= B_j(N, r^c_{N,v,S}) - B_j(N, r^c_{N,v,S_{-(h,j)}}) = \beta^c_j(N,v,S) - \beta^c_j(N,v,S_{-(h,j)}).$$

$\square$

Appendix C: Logical independence

We show logical independence of the axioms of Theorem 3.5 by the following nine solutions for games with a hierarchical permission structure.

1. The solution f given by $f_i(N,v,S) = 0$ for all $i \in N$ satisfies all axioms of Theorem 3.5 except one player efficiency.

2. The solution f given by $f_{i_0}(N,v,S) = v(\{i_0\})$, and $f_i(N,v,S) = 0$ for all $i \in N \setminus \{i_0\}$ satisfies all axioms of Theorem 3.5 except vertical split neutrality. (Remember that i_0 denotes the top-player in (N,S).)

3. The solution f given by $f(N,v,S) = \varphi^d(N, \sum_{i \in N} v(\{i\})u_{\{i\}}, S)$ satisfies all axioms of Theorem 3.5 except horizontal split neutrality.

4. For $S \in \mathcal{S}_H^N$ and $T \subset N$ let $\mathcal{Z}_T^S = \{Z \in \mathcal{S}_H^N \mid Z^{-1}(i) = S^{-1}(i)$ for all $i \in N \setminus \alpha_S(T)$, $Z^{-1}(i) \subset S^{-1}(i)$ for all $i \in \alpha_S(T)$, and $|Z^{-1}(i)| = 1$ for all $i \in \alpha_S(T) \setminus \{i_0\}\}$ where again $\alpha_S(T) = T \cup \widehat{S}^{-1}(T)$. The solution f given by $f(N,v,S) = \sum_{T \subset N} \sum_{Z \in \mathcal{Z}_T^S} \varphi_i^d(N, \Delta_{(N,v)}(T)u_T, Z)$ satisfies all axioms of Theorem 3.5 except power split neutrality.

5. For game (N,v) define $d(N,v) = \max\{\Delta_{(N,v)}(T) \mid T \subset N\}$, where $\Delta_{(N,v)}(T)$ are the Harsanyi dividends, and $D(N,v) = \{T \subset N \mid \Delta_{(N,v)}(T) = d(N,v)\}$. The solution f given by $f(N,v,S) = \varphi^d(N, \sum_{T \in D(N,v)} \Delta_{(N,v)}(T)u_T, S)$ satisfies all axioms of Theorem 3.5 except additivity.

6. The solution f given by $f_i(N,v,S) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies all axioms of Theorem 3.5 except the inessential player property.

7. The solution f given by $f_{i_0}(N,v,S) = v(N)$ and $f_i(N,v,S) = 0$ for all $i \in N \setminus \{i_0\}$ satisfies all axioms of Theorem 3.5 except the necessary player property.

8. The solution f given by $f(N,v,S) = Sh(N,v)$ satisfies all axioms of Theorem 3.5 except weak structural monotonicity.

9. The conjunctive Shapley permission value φ^c satisfies all axioms of Theorem 3.5 except disjunctive fairness. □

Logical independence of the axioms of Theorem 3.6 can be shown by the above solutions 1, 2, 6, 7 and 8, using φ^c instead of φ^d in solutions 3, 4 and 5, and replacing φ^c by φ^d in solution 9.

Similarly, logical independence of the axioms of Theorem 4.4 can be shown by the above solutions 1, 2, 6, 7 and 8, using β^d instead of φ^d in solutions 3, 4 and 5, and replacing φ^c by β^c in solution 9.

Finally, logical independence of the axioms of Theorem 4.5 can be shown by the above solutions 1, 2, 6, 7 and 8, using β^c instead of φ^d in solutions 3, 4 and 5, and replacing φ^c by β^d in solution 9.

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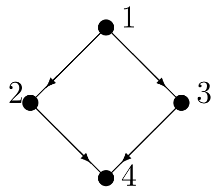

Figure 1: Permission structure S from Example 2.1

Figure 2: An illustration of permission structures S and $S^{V(h,h)}$ as described in Axiom 3.1

Figure 3: An illustration of permission structures S and $S^{H(h,j)}$ as described in Axiom 3.2

Figure 4: An illustration of permission structure S_{-h}

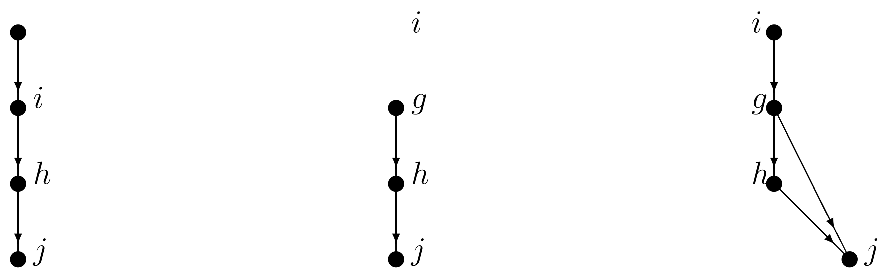

Figure 5: An illustration of permission structures S, S' and S'' of the proof of Lemma B.2 case (iii)

Figure 6: An illustration of permission structures S' and S'' of the proof of Lemma B.3 case (ii), with $h \in S(g)$ (So, $h = g$)

Figure 7: An illustration of permission structures S' and S'' of the proof of Lemma B.3 case (i) with $h \in S(i_0)$ and $j \in S(h)$